

# Field-parametrization dependence of Dirac's method for constrained Hamiltonians with first-class constraints: failure or triumph? Non-covariant models

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## Abstract

We argue that the field-parametrization dependence of Dirac's procedure, for Hamiltonians with first-class constraints not only preserves covariance in covariant theories, but in non-covariant gauge theories it allows one to find the natural field parametrization in which the Hamiltonian formulation automatically leads to the simplest gauge symmetry.

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## I. INTRODUCTION

In physics, one cannot overstate the importance of gauge invariance; all theories of fundamental interactions have this property. The description of gauge symmetries in the Lagrangian and Hamiltonian formulations has attracted considerable interest and has a long history. For covariant theories, natural variables are those that are true tensors with respect to Lorentz or general coordinate transformations. If such variables are used, then the theory is manifestly covariant. In addition, the Hamiltonian analysis and the subsequent restoration of the gauge transformations from the first-class constraints (Dirac's conjecture [1]) leads to results equivalent to those found using the Lagrangian approach (e.g. the Maxwell and Yang-Mills theories, and General Relativity (GR) with the natural variable: i.e. metric tensor). Non-covariant changes of field parametrisation produce different results (e.g. gauge symmetries, algebra of constraints, etc.), and the most prominent example is the ADM parametrisation, which was analysed in [2–4]. The field-parametrisation dependence of the Dirac procedure for systems with first-class constraints (i.e. gauge invariant systems) allows one to single out the one particular parametrisation that is consistent with covariance, and thus find the Lagrangian formulation for which the gauge invariance is related to Noether's differential identities (DIs) – a linear combination of Euler-Lagrange derivatives (ELDs). Such DIs and the corresponding gauge transformations are covariant.

Working with the Lagrangians of either non-covariant systems or covariant systems without the restrictions imposed by covariance, one may take various linear combinations of the known DIs to construct many more DIs that describe different gauge transformations, as well as any field parametrisation (invertible change of variables) can be used to rewrite the DIs. Therefore, many gauge symmetries can be easily constructed at the Lagrangian level, in all conceivable parametrisations. According to Noether's theorem [5, 6], an important characteristic of a gauge theory is its maximum number of independent DIs, and if the value of this maximum number is preserved many different combinations of the DIs may be used. At the Lagrangian level, all of the different gauge symmetries, and all of the different field parametrisations are independent – each symmetry can be written in any parametrisation, and in each parametrisation any symmetry can be described. Of course these symmetries can have different properties – to form or not to form a group. In the simplest case (such as the Maxwell theory) a commutator of two consecutive gauge transformations is zero; in other

cases it may be characterised by a field-independent structure constant or a field-dependent structure function. The field-parametrisation dependence of Hamiltonian formulations of gauge theories causes there to be a different relationship among the parametrisations and symmetries. All parametrisations and symmetries obtained in the Lagrangian approach can also be described using Hamiltonian methods; but unlike the Lagrangian approach, the symmetries become uniquely related to a chosen field parametrisation. Is this parametrisation dependence a failure or a triumph of Dirac’s method? To answer this question, this dependence should be analysed and understood.

If all possible symmetries (and gauge transformations) can be classified on the basis of some criteria (e.g. do they or do they not form a group? The simplicity of commutators, etc.)<sup>1</sup>, then the Hamiltonian methods allow one to find the corresponding parametrisation through the parametrisation dependence of the Hamiltonian procedure for constrained systems. Often this property is considered a weakness of the Hamiltonian approach for constrained systems, and various attempts to modify Dirac’s procedure or the more revolutionary proposal that one may not consider “the Dirac approach as fundamental and undoubted” appear in the literature (e.g. [7]). The main rationalisation, at the root of such radical suggestions, is that in particular parametrisations, Dirac’s method refuses to produce the “correct” or “expected” symmetries; but in our view this behaviour is an important property that, in the case of covariant theories, rules out non-covariant changes of field variables [9].

The goal of our paper is twofold. Firstly, to illustrate the connection of field parametrisations to the associated gauge symmetries for the Lagrangian and Hamiltonian formulations. A change of parametrisation is usually discussed in conjunction with GR where the calculations are very involved, and where considering different parametrisations would be technically very difficult. By using simple examples we may better elucidate the results. Secondly, to demonstrate the utility of the field-parametrisation dependence for the case of non-covariant models, where the most natural parametrisation for particular models can be found in an algorithmic way by using Hamiltonian analysis.

In the next Section we consider the Henneaux-Teitelboim-Zanelli (HTZ) [10] model and

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<sup>1</sup> Merely calling one symmetry ‘correct’ and another ‘incorrect’ (as for example, in [7]) should not be used as a criterion [8].

demonstrate a general procedure to find a natural parametrisation when using the Hamiltonian approach. In Section 3 the Isotropic Cosmological Model [7] is considered as another example of the application of this procedure, and the effect of a change of field parametrisation at the Lagrangian and Hamiltonian levels is discussed. The results are summarised in Conclusion. In Appendix A, the restoration of gauge invariance from the first-class constraints of the Hamiltonian formulation of the HTZ model in the original parametrisation is performed using the Castellani procedure [11].

## II. THE HENNEAUX-TEITELBOIM-ZANELLI MODEL

To illustrate the role played by the field-parametrisation dependence of Dirac's procedure for Hamiltonians with first-class constraints in finding the natural (simplest) gauge symmetry in the Lagrangian parametrisation, we shall first consider a simple model that was introduced and discussed in [10] and in the book [12] (see p. 88). Despite the simplicity of this model, the application of the Dirac procedure leads to tertiary constraints. The analog in field theory, in the sense of the appearance of tertiary constraints, is found in the Hamiltonian analysis [13] of the affine-metric formulation of GR [14, 15].

The Lagrangian of the simple model of [10, 12] is

$$L = \frac{1}{2} [(\dot{q}_2 - e^{q_1})^2 + (\dot{q}_3 - q_2)^2]. \quad (1)$$

The majority of readers will note that the variable  $q_1$  enters the Lagrangian only once, and in such a form that one must wonder if it would not have been better to redefine  $e^{q_1}$  as a new simple variable at the outset, instead of keeping it as a function. Such a field parametrisation would be more natural for this Lagrangian; and one might doubt that the form given by (1) is the natural choice to make for any analysis, or if it could lead to any insight in the study of the gauge symmetries of such a model. By using a procedure that relies upon the field-parametrisation dependence of Dirac's Hamiltonian formulation, we show that such suspicions are correct. Further, this procedure is a general one, and it can be applied equally well to more complicated Lagrangians.

### A. The Hamiltonian Formulation of the HTZ Model

The Hamiltonian formulation of (1) was considered in [10] by using the Dirac procedure. We elaborate upon this example because we shall need the details for our subsequent discussion of the model of [10, 12]. First, perform the Legendre transformation,

$$H_T = p_1 \dot{q}_1 + p_2 \dot{q}_2 + p_3 \dot{q}_3 - L ,$$

where  $p_i = \frac{\partial L}{\partial \dot{q}_i}$  are momenta conjugate to  $q_i$ . After expressing velocities (solvable) in terms of momenta, the total Hamiltonian follows:

$$H_T = p_1 \dot{q}_1 + e^{q_1} p_2 + q_2 p_3 + \frac{1}{2} (p_2)^2 + \frac{1}{2} (p_3)^2 . \quad (2)$$

The time development of the primary constraint,

$$\phi_1 \equiv p_1 , \quad (3)$$

leads to the secondary constraint,

$$\dot{\phi}_1 = \{\phi_1, H_T\} = -e^{q_1} p_2 \equiv \phi_2 ; \quad (4)$$

and, in turn, the time development of the secondary yields the tertiary,

$$\dot{\phi}_2 = \{\phi_2, H_T\} = e^{q_1} p_3 \equiv \phi_3 . \quad (5)$$

The time development of the tertiary constraint is proportional to itself,

$$\dot{\phi}_3 = \{\phi_3, H_T\} = \dot{q}_1 \phi_3 . \quad (6)$$

Thus no new constraints appear, and closure is attained. The algebra of constraints is simple and has the following Poisson Brackets (PBs):

$$\{\phi_1, \phi_2\} = -\phi_2 , \quad \{\phi_1, \phi_3\} = -\phi_3 , \quad \{\phi_2, \phi_3\} = 0. \quad (7)$$

Therefore all of the constraints are first-class. According to the Dirac conjecture [1], a knowledge of all first-class constraints is sufficient for finding the gauge transformations.

Using the method<sup>2</sup> proposed in [10], the transformations were restored by constructing a gauge generator that allows one to find the transformations of phase-space variables. After

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<sup>2</sup> According to the authors of [10] “our formalism is capable of handling such cases without difficulties”; but the Castellani algorithm [11] also leads to the same gauge transformations (see Appendix A of this paper).

the elimination of momenta (i.e. the back-substitution of momenta in terms of velocities) the Lagrangian transformations were found in [10, 12]:

$$\delta q_1 = \varepsilon + 2\dot{q}_1\varepsilon + \ddot{q}_1\varepsilon + (\dot{q}_1)^2\varepsilon, \quad \delta q_2 = e^{q_1}(\dot{\varepsilon} + \dot{q}_1\varepsilon), \quad \delta q_3 = e^{q_1}\varepsilon \quad (8)$$

(see also [16], where they were obtained by a different method, and Appendix A, where these transformations are derived using the Castellani algorithm).

Equation (8) describes gauge transformations that uniquely follow from Dirac's Hamiltonian analysis of Lagrangian (1); and it correctly reproduces the symmetry of the Lagrangian, which in this case can be easily and directly checked by performing a variation of the Lagrangian under transformations (8):

$$\delta L = 0. \quad (9)$$

## B. The Lagrangian Formulation of the HTZ Model

Noether's second theorem [5], which we shall need in our discussion, can be used also for more complicated models. If the transformations are known, then we can restore the DI – a combination of Euler-Lagrange derivatives that is identically equal to zero (“off-shell”, i.e. without imposing equations of motion,  $ELD = 0$ ). For example, a restoration of the DI from a known transformation was performed by Schwinger [17] in a discussion of the Einstein-Cartan action (see also [18]); but the first appearance of such an approach can at least be traced back to the earlier work of Rosenfeld [19] (see Eqs. (71, 72))<sup>3</sup>. Consider

$$\int [E_{(q_1)}\delta q_1 + E_{(q_2)}\delta q_2 + E_{(q_3)}\delta q_3] dt = \int I\varepsilon dt, \quad (10)$$

where  $I$  is a DI and  $E_{(q_i)}$  are ELDs for (1) (see Eqs. (41-43) of [16]):

$$E_{(q_1)} = \frac{\delta L}{\delta q_1} = -\dot{q}_2 e^{q_1} + e^{2q_1}, \quad (11)$$

$$E_{(q_2)} = \frac{\delta L}{\delta q_2} = -\dot{q}_3 + q_2 - \ddot{q}_2 + e^{q_1}\dot{q}_1, \quad (12)$$

and

$$E_{(q_3)} = \frac{\delta L}{\delta q_3} = -\ddot{q}_3 + \dot{q}_2. \quad (13)$$

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<sup>3</sup> Thanks to Salisbury [20], the paper became available to non-German readers.

By substituting transformation (8) into the left hand side of (10), and singling out the gauge parameter, one obtains the corresponding DI

$$I = \ddot{E}_{(q_1)} - 2\dot{E}_{(q_1)}\dot{q}_1 - E_{(q_1)}\ddot{q}_1 + (\dot{q}_1)^2 E_{(q_1)} - e^{q_1}\dot{E}_{(q_2)} + e^{q_1}E_{(q_3)} \equiv 0, \quad (14)$$

an identity that can be verified by the substitution of (11)-(13). It is a straightforward procedure to handle expressions of any complexity since terms of different type can be considered separately. Direct variation, where in general we must combine terms under the total derivative(s) is more difficult; in more complicated theories, one might not be able to recognise the combinations of terms that form a total derivative (but not in the case of (9)).

Let us check the properties of transformations (8) by calculating the commutator such that

$$[\delta_1, \delta_2] \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = (\delta_1\delta_2 - \delta_2\delta_1) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \delta_{[1,2]} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}. \quad (15)$$

For all fields  $q_i$  the calculation gives

$$\varepsilon_{[1,2]} = \varepsilon_2\ddot{\varepsilon}_1 - \varepsilon_1\ddot{\varepsilon}_2 + 2\dot{q}_1(\varepsilon_2\dot{\varepsilon}_1 - \varepsilon_1\dot{\varepsilon}_2). \quad (16)$$

A field appears in the definition of parameter  $\varepsilon_{[1,2]}$ ; this might lead to a possible problem in which the transformations do not form a group. It is already an indication that some simpler transformations might exist for this Lagrangian, as in the example of the Maxwell Lagrangian, which is quadratic in fields; Lagrangian (1) would also be quadratic in fields and their derivatives if another parametrisation were to be considered.

### C. Finding a Simpler Gauge Symmetry

Is it possible to find another simpler symmetry? And might this simple symmetry have a commutator of gauge transformations equal to zero? One DI that uniquely follows from the Hamiltonian analysis of (1) is known, i.e. (14); and by using this DI we can construct another DI, which according to the converse of Noether's second theorem [5] would give another symmetry with a simpler commutator, either equal to zero or without a field-dependent structure function. Note that Noether's second theorem refers to the maximum number of independent DIs. It is obvious that if one of the ELDs enters the DI in such a form

that it has no field-dependent coefficients, then the corresponding transformations would depend only on the gauge parameter, and the commutator of such transformations would be zero. This result is the simplest possible, and for such a rudimentary model as HTZ with Lagrangian (1) (which is quadratic if non-linearity is not introduced by some weird choice of parametrisation) it is the result to expect. This subject is certainly something to explore. The analogy in field theory is the Maxwell electrodynamics, with a Lagrangian quadratic in velocities, where the commutator of the gauge transformations is zero (of course if someone did not introduce a non-covariant change of variables). Inspecting (14), we can see that two of the three ELDs appear once, and it is not difficult to eliminate their field-dependent coefficients by performing a multiplication of DI (14) by  $e^{-q_1}$ ,

$$\tilde{I} = e^{-q_1} I = e^{-q_1} \ddot{E}_{(q_1)} - 2e^{-q_1} \dot{q}_1 \dot{E}_{(q_1)} - e^{-q_1} \ddot{q}_1 E_{(q_1)} + e^{-q_1} (\dot{q}_1)^2 E_{(q_1)} - \dot{E}_{(q_2)} + E_{(q_3)} \equiv 0 \quad (17)$$

(even though the DI is modified, it is still a DI). This short cut was based on the simple analysis of the DI for this particular model. But in general, if it is unclear how to modify a DI, then one can multiply it by some function of the fields of the model to find the transformations, which correspond to the DI; one may then solve for the function under the condition that it makes the commutator of gauge transformations equal to zero. If it is not possible to find such a solution, then a function that preserves the Jacobi identity may be sought (this approach will be used in the next Section).

Two of the ELDs in (17) now have field-independent coefficients. Let us seek the corresponding transformations by performing the operation (10) that leads to (14) in inverse order. Note that we are working with the same parametrisation as before (the fields are unchanged), but the DI is different, and so are the corresponding transformations:

$$\tilde{\delta}q_1 = e^{-q_1} \ddot{\eta}, \quad \tilde{\delta}q_2 = \dot{\eta}, \quad \tilde{\delta}q_3 = \eta. \quad (18)$$

In (18) we call the gauge parameter by  $\eta$  (which is field-independent as was  $\varepsilon$  before), and the transformation by  $\tilde{\delta}$ .

Some terms cancel out, when performing the calculation with (17), which suggests a simpler way of presenting the DI (of course, some might have recognised such a simplification at the previous stage, but to do so for a more complicated model can be very difficult; further, our goal is to demonstrate an algorithm that automatically produces this simpler

presentation). From transformations (18), a short form of DI (17) follows –

$$\tilde{I} = \frac{d^2}{dt^2} (e^{-q_1} E_{(q_1)}) - \dot{E}_{(q_2)} + E_{(q_3)} \equiv 0. \quad (19)$$

It is obvious that for all of the fields, the commutator of transformations (18) is zero, i.e.

$$\left[ \tilde{\delta}_1, \tilde{\delta}_2 \right] \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = 0. \quad (20)$$

Note that this is *the same parametrisation* as that of the original Lagrangian, but *the symmetry is now different*, i.e.  $\tilde{\delta}$  (compare (8) and (18)); further, transformations (8) are complicated, and the others, (18), are the simplest possible (zero commutators). We can find more new DIs for this parametrisation, obtain various transformations, and check their group properties; but this one, (19), is the simplest.

#### D. General Procedure of Finding the Simplest Parametrisation

If one gauge symmetry is already known for the Lagrangian in the parametrisation considered, then additional symmetries can be constructed. For example, a symmetry, which might not be of the simplest form, may be obtained from the Hamiltonian formulation. We can then construct many other symmetries and find the simplest one. Is it possible to further simplify (18) and (19), or might there be another parametrisation that, in addition, will lead to this simple symmetry in the Hamiltonian approach, and so provide the canonical, or the simplest parametrisation for this model? By its construction, the Hamiltonian method leads to a symmetry of the Lagrangian, but only one for each particular parametrisation (some freedom might remain, and we will discuss this later) with a DI that always starts from the highest-order time derivative of some ELD without a field-dependent coefficient (e.g. the Maxwell and Yang-Mills theories, metric gravity and DI (14) for (1)). Let us call such a ELD *the leading ELD*. This is not the case for DI (19) in this parametrisation: the leading ELD enters (17) with the coefficient  $e^{-q_1}$ , and that is why the Hamiltonian method produces another symmetry that corresponds to DI (14) (i.e.  $I = \ddot{E}_{(q_1)} + \dots$ ); but one may look for another parametrisation to obtain this same, simple symmetry. It is easy to find a new parametrisation for this model because the leading ELD in (19) and its coefficient share the same field; therefore, we can just look for a new parametrisation of this field without

having to consider a mixture of all of the variables of the model. The general procedure of changing a DI under a change of variables was discussed in Conclusion of [9].

We can attempt to change only one variable. Consider some invertible function

$$q_1 = q_1(q);$$

the relation among the ELDs for such a change is,

$$E_{(q)} = \frac{\delta L(q_1 = q_1(q), q_2, q_3)}{\delta q} = \frac{\delta L}{\delta q_1} \frac{dq_1}{dq} = E_{(q_1)} \frac{dq_1}{dq}.$$

In equation (19), the removal of the field-dependent coefficient requires:

$$\frac{dq_1}{dq} = e^{-q_1}.$$

And by solving this ordinary differential equation (ODE), we obtain the parametrisation

$$q = e^{q_1}, \tag{21}$$

which yields the simplest possible Noether DI,

$$\tilde{I} = \ddot{E}_{(q)} - \dot{E}_{(q_2)} + E_{(q_3)} \equiv 0, \tag{22}$$

and the gauge transformations

$$\tilde{\delta}q = \ddot{\eta}, \quad \tilde{\delta}q_2 = \dot{\eta}, \quad \tilde{\delta}q_3 = \eta. \tag{23}$$

Note that (23) and (18) represent the same symmetry, but for different parametrisations; this situation is distinct from having two different symmetries (i.e. (8) and (18)) for the same parametrisation. Further, DI (19) is the same as DI (22); thus we have arrived at a natural, or canonical, parametrisation of the initial Lagrangian – a parametrisation that is consistent with the one that might have been the more logical choice at the outset. In this particular parametrisation, the Lagrangian (1) is

$$\tilde{L} = \frac{1}{2} [(\dot{q}_2 - q)^2 + (\dot{q}_3 - q_2)^2]. \tag{24}$$

### E. The Hamiltonian Analysis in the New, Canonical Parametrisation

Let us return to the Hamiltonian formulation. If one starts from Lagrangian (24), then the total Hamiltonian is

$$\tilde{H}_T = p\dot{q} + qp_2 + q_2p_3 + \frac{1}{2}(p_2)^2 + \frac{1}{2}(p_3)^2 \tag{25}$$

with the primary constraint

$$\tilde{\phi}_1 \equiv p,$$

where  $p$  is a momentum conjugate to  $q$ . The secondary,

$$\left\{ \tilde{\phi}_1, H_T \right\} = -p_2 \equiv \tilde{\phi}_2,$$

and tertiary,

$$\left\{ \tilde{\phi}_2, H_T \right\} = p_3 \equiv \tilde{\phi}_3,$$

constraints follow from the conservation in time of the primary and secondary constraints. Since there are no new constraints, as  $\left\{ \tilde{\phi}_3, H_T \right\} = 0$ , the Dirac procedure closes on the tertiary constraint (as in the case for Lagrangian (1)). Unlike (7), all of the PBs among the constraints are zero, i.e.

$$\left\{ \tilde{\phi}_1, \tilde{\phi}_2 \right\} = \left\{ \tilde{\phi}_1, \tilde{\phi}_3 \right\} = \left\{ \tilde{\phi}_2, \tilde{\phi}_3 \right\} = 0, \quad (26)$$

since all the constraints are pure momenta.

For the Lagrangians in two distinct parametrisations, (1) and (24), the Dirac method gives two different symmetries, (8) and (23). What is the relationship between the Hamiltonians of the two different parametrisations? For these two parametrisations, it is not difficult to find the canonical transformations between their phase-space variables:

$$q = e^{q_1}, \quad p = e^{-q_1} p_1, \quad (27)$$

which satisfy the PBs

$$\{q, p\}_{q_1, p_1} = \{q_1, p_1\}_{q, p} = 1.$$

Only one field and its momentum were changed, and the rest of the variables remain the same. This change of variables, (27), is canonical.

For constrained systems, the ordinary canonicity condition [21] is necessary, but not sufficient to have equivalent Hamiltonians (i.e. that lead to the same gauge symmetry); the algebra of constraints must also be preserved, which is not the case for the two Hamiltonians (2) and (25) (compare with (7) and (26)). This additional condition for the two Hamiltonians with first-class constraints was conjectured in [22], based on a comparison of the Hamiltonian formulations of GR due to Pirani, Schild, Skinner (PSS) [3], and due to Dirac [4], both of which lead to diffeomorphism invariance [2, 23].

In [10, 12] it was noticed that if a gauge parameter is redefined as  $\eta = \eta(q_1, \varepsilon) = e^{q_1} \varepsilon$ , then one can present (8) in the short form (19). This result is consistent with the statement made in [10]: “... the redefinition of the gauge parameter  $\varepsilon e^{q_1} \rightarrow \eta$  simplify the form of the gauge transformations ...”. If this were only a matter of “form” (i.e. just a shorthand notation), then it would be the same symmetry; but if we consider (8) and treat  $\eta$  as a field-independent parameter, then it is a different transformation, as demonstrated by commutators (7) and (26). Any field-dependent redefinition of parameters is a new symmetry, which is clear from the DI. Such a redefinition is equivalent to the multiplication of a DI by some combination of fields; it is a new DI that produces a new symmetry. For example, consider DI (14); multiplying it by a field-dependent parameter is in effect a construction of a new DI, as shown in (17). DIs (14) and (17) correspond to two different symmetries, one is complicated and the other has a zero commutator (as in the Maxwell theory). The latter symmetry cannot be obtained by the Dirac Hamiltonian method – for this choice of parametrisation the complicated one follows uniquely.

To illustrate this difference, let us consider one of transformations (23)

$$\tilde{\delta}q = \ddot{\eta};$$

and using change of variables (21), find the transformation of  $q_1$

$$\tilde{\delta}q_1 = \tilde{\delta}(\ln q) = \frac{1}{q} \tilde{\delta}q = e^{-q_1} \ddot{\eta}, \quad (28)$$

which is a symmetry with a zero commutator (if the parameter  $\eta$  is field-independent). Transformation (28) differs from ones of (8) that follow from Hamiltonian (2) and give commutator (15) with parameter (16).

The “field dependent redefinition of gauge parameter” is only useful as a formal trick to simplify some calculations, including the calculation of commutators and double commutators as in [9, 24]. If the parameter were to depend on the fields, then it would also be affected by the change of variables. If we were to calculate a commutator, the presence of field-dependent parameters would change the result; therefore, the formal connection (e.g.  $\eta = e^{q_1} \varepsilon$ ) is actually misleading because the corresponding gauge symmetries are in fact different.

Recall the statement at the beginning of Section 4.4 of [10]: “The form of gauge transformations is not unique. One can redefine the gauge parameters ...”. If a gauge parameter

is redefined in the same parametrisation, then it will correspond to a different symmetry, and it is not a difference in “form”. The form of a given gauge transformation is not unique since different parametrisations may be used without causing a change of symmetry; but if the gauge parameters are redefined it is not a different form of the same symmetry, it is a different symmetry.

## F. Change of Non-Primary Variables

Let us discuss the conditions under which the form of gauge transformations may be changed, but the symmetry remains the same (we do not change the gauge parameters or multiply the DI by some function; to do so would lead to another symmetry). Any change of parametrisation will make DI (22), which is the simplest possible, more complicated; but what will be the effect of such a change on the gauge symmetry? In the Hamiltonian approach, we always obtain a symmetry with a corresponding DI such that the leading ELD (the one with the highest order of time derivative) has no field-dependent coefficient. Let us consider changes of parametrisation that do not affect the variables that correspond to the leading ELD. We shall call them *the primary variables*. They also play an important role in the Hamiltonian analysis [13, 25]: *the primary variables* are those which conjugate momenta enter the primary first-class constraints. The other variables, which can be eliminated (along with their conjugate momenta) in the Hamiltonian reduction by solving the second-class constraints, are called *second-class variables* (or *secondary, non-primary variables*). In the literature the primary variables are more widely referred to as ‘gauge’ variables, which is a somewhat confusing name since in a gauge invariant system all variables are involved in a gauge symmetry, i.e. a Lagrangian is invariant under the gauge transformations of *all* the variables in configurational space.

One may wonder if there can still be some freedom to choose the parametrisation, and to do so in a way that is consistent with both the Lagrangian and Hamiltonian approaches? We will consider this question in general, but take the HTZ model as an example. There are two non-leading ELDs in (22), which give more flexibility; any invertible change of the corresponding non-leading variables, including their mixture, will not affect the gauge transformations, thus keeping the Hamiltonians equivalent; the symmetry remains the same, even though the transformations are of a different form.

Let us define the transformations,

$$q_2 = q_2(v, w), \quad q_3 = q_3(v, w), \quad (29)$$

and their inverse,

$$\nu = v(q_2, q_3), \quad w = w(q_2, q_3),$$

to which the condition of invertibility applies,

$$J = \det \begin{bmatrix} \frac{\partial q_2}{\partial v} & \frac{\partial q_3}{\partial v} \\ \frac{\partial q_2}{\partial w} & \frac{\partial q_3}{\partial w} \end{bmatrix} \neq 0. \quad (30)$$

Consider the Lagrangian

$$L(q, q_2, q_3) = L(q, q_2(v, w), q_3(v, w)) = L(q, v, w) = L(q, v(q_2, q_3), w(q_2, q_3));$$

performing this change for DI (23), we must express the ELDs of the initial parametrisation in terms of the ELDs and fields of the new one. We have

$$E_{(v)} = \frac{\delta L}{\delta v} = \frac{\delta L}{\delta q_2} \frac{\partial q_2}{\partial v} + \frac{\delta L}{\delta q_3} \frac{\partial q_3}{\partial v} = E_{(q_2)} \frac{\partial q_2}{\partial v} + E_{(q_3)} \frac{\partial q_3}{\partial v} \quad (31)$$

and

$$E_{(w)} = \frac{\delta L}{\delta w} = \frac{\delta L}{\delta q_2} \frac{\partial q_2}{\partial w} + \frac{\delta L}{\delta q_3} \frac{\partial q_3}{\partial w} = E_{(q_2)} \frac{\partial q_2}{\partial w} + E_{(q_3)} \frac{\partial q_3}{\partial w}; \quad (32)$$

as in [9], we solve (31) and (32) to find the relation between the Euler-Lagrange derivatives in this new parametrisation, i.e.

$$E_{(q_2)} = \frac{E_{(v)} \frac{\partial q_3}{\partial w} - E_{(w)} \frac{\partial q_3}{\partial v}}{J}, \quad E_{(q_3)} = \frac{\frac{\partial q_2}{\partial v} E_{(w)} - \frac{\partial q_2}{\partial w} E_{(v)}}{J}.$$

After performing the substitution into (22), we obtain the DI in terms of the new variables and corresponding ELDs:

$$\tilde{I} = \ddot{E}_{(q)} - \dot{E}_{(q_2)} + E_{(q_3)} = \ddot{E}_{(q)} - \frac{d}{dt} \left( \frac{E_{(v)} \frac{\partial q_3}{\partial w} - E_{(w)} \frac{\partial q_3}{\partial v}}{J} \right) + \frac{\frac{\partial q_2}{\partial v} E_{(w)} - \frac{\partial q_2}{\partial w} E_{(v)}}{J};$$

and from this result, the gauge transformations of the fields follow:

$$\tilde{\delta} q = \ddot{\eta}, \quad \tilde{\delta} v = \frac{\frac{\partial q_3}{\partial w}}{J} \dot{\eta} + \frac{-\frac{\partial q_2}{\partial w}}{J} \eta, \quad \tilde{\delta} w = \frac{-\frac{\partial q_3}{\partial v}}{J} \dot{\eta} + \frac{\frac{\partial q_2}{\partial v}}{J} \eta. \quad (33)$$

Compared with transformations (23), the new *form* (i.e. (33)) is very different, in general.

To show the equivalence of the two sets of transformations, (23) and (33), let us find the transformation of  $q_2$ :

$$\tilde{\delta} q_2 = \frac{\partial q_2}{\partial v} \tilde{\delta} v + \frac{\partial q_2}{\partial w} \tilde{\delta} w. \quad (34)$$

After substitution of (33) into (34) and subsequent simplification, we return to the transformations (23). The same result can be found for  $q_3$ . Since we did not perform a multiplication of the DI by a function of fields, the change of variables in (29) does not affect the gauge invariance. This behaviour is expected. If the DI is modified by changing to another parametrisation in the manner just described, then the symmetry is unchanged. This is a true difference in the “form” of the one (the same) symmetry, i.e. the form of the gauge transformations is changed, but the commutators among the gauge transformations of the fields remain the same.

The conjecture was made in [22] that to have the same symmetry between different parametrisations at the Hamiltonian level, the canonicity of the phase-space variables is not sufficient, the whole algebra of the PBs of constraints should be unchanged.

### G. Change of Non-Primary Variables at the Hamiltonian Level

Let us now investigate the effect of change of variables (29) at the Hamiltonian level, but without starting from a Lagrangian that is written in new variables (such calculations are not difficult); instead let us perform a change of phase-space variables in total Hamiltonian (25).

At the Hamiltonian level, we can find canonical transformations for change (29) (e.g. see [21]):

$$\begin{aligned} p_2 \dot{q}_2 + p_3 \dot{q}_3 &= p_2 \left( \frac{\partial q_2}{\partial v} \dot{v} + \frac{\partial q_2}{\partial w} \dot{w} \right) + p_3 \left( \frac{\partial q_3}{\partial v} \dot{v} + \frac{\partial q_3}{\partial w} \dot{w} \right) = \\ &= \left( p_2 \frac{\partial q_2}{\partial v} + p_3 \frac{\partial q_3}{\partial v} \right) \dot{v} + \left( p_2 \frac{\partial q_2}{\partial w} + p_3 \frac{\partial q_3}{\partial w} \right) \dot{w} = \pi_v \dot{v} + \pi_w \dot{w}; \end{aligned}$$

and we have

$$\pi_v = p_2 \frac{\partial q_2}{\partial v} + p_3 \frac{\partial q_3}{\partial v}$$

and

$$\pi_w = p_2 \frac{\partial q_2}{\partial w} + p_3 \frac{\partial q_3}{\partial w}.$$

We may express the momenta of the original formulation in terms of the new variables

$$p_2 = \frac{\pi_v \frac{\partial q_3}{\partial w} - \pi_w \frac{\partial q_3}{\partial v}}{J}, \quad p_3 = \frac{\frac{\partial q_2}{\partial v} \pi_w - \frac{\partial q_2}{\partial w} \pi_v}{J},$$

where  $J$  is given by (30). We now have a canonical relationship, and the only non-zero PBs are,

$$\{q_2, p_2\}_{v,w,\pi_v,\pi_w} = \{q_3, p_3\}_{v,w,\pi_v,\pi_w} = 1; \quad (35)$$

the change of variables is canonical, in the ordinary sense, because of the invertibility of transformations (29). Performing this transformation in Hamiltonian (25), only the so-called canonical part is affected by such a change<sup>4</sup>,

$$H_T(p, q, v, \pi_v, w, \pi_w) = p\dot{q} + H_c, \quad (36)$$

$$H_c = q \frac{\pi_v \frac{\partial q_3}{\partial w} - \pi_w \frac{\partial q_3}{\partial v}}{J} + q_2(v, w) \frac{\frac{\partial q_2}{\partial v} \pi_w - \frac{\partial q_2}{\partial w} \pi_v}{J} + \frac{1}{2} \left( \frac{\pi_v \frac{\partial q_3}{\partial w} - \pi_w \frac{\partial q_3}{\partial v}}{J} \right)^2 + \frac{1}{2} \left( \frac{\frac{\partial q_2}{\partial v} \pi_w - \frac{\partial q_2}{\partial w} \pi_v}{J} \right)^2.$$

The time development of primary constraint  $p$  leads to the secondary constraint,

$$\dot{p} = \{p, H_T\} = -\frac{\pi_v \frac{\partial q_3}{\partial w} - \pi_w \frac{\partial q_3}{\partial v}}{J} = \hat{\phi}_2,$$

and the time derivative of the secondary constraint yields the tertiary one,

$$\{\hat{\phi}_2, H_T\} = \frac{\frac{\partial q_2}{\partial v} \pi_w - \frac{\partial q_2}{\partial w} \pi_v}{J} = \hat{\phi}_3.$$

Because of canonicity relation (35), these constraints have the same algebra of PBs as (26); and the two Hamiltonians, (25) and (36), describe the same symmetry. Based on these constraints, generators can be built, Hamiltonian gauge transformations found, and the Lagrangian transformations (33) derived.

## H. Discussion

We demonstrated, using the HTZ model, that the field-parametrisation dependence of the Dirac procedure allows one to find, for a Lagrangian written in some parametrisation, the canonical parametrisation that leads to the simplest gauge transformations with trivial group properties. The procedure described in this Section can be used to treat more complicated models or theories, and the obvious and natural results obtained for the HTZ model illustrate

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<sup>4</sup> The name “canonical” for the part without primary constraints conveys the idea that it is possible to perform canonical transformations as in ordinary unconstrained Hamiltonians, if the primary variables are not involved in such changes. This is true even in this example where the whole canonical part is proportional to the constraints (secondary and tertiary).

the value of such an approach. Canonical symmetry is a unique property of the Lagrangian, but the choice of parametrisation is not; any canonical changes of field variables that preserve the algebra of constraints keeps the symmetry in tact. Changes of variables that involve primary variables, or their mixture with the non-primary variables, will lead to another symmetry, and thus cannot be performed at the Hamiltonian level (such changes are non-canonical). We shall illustrate this behaviour with the example in the next Section.

### III. ISOTROPIC COSMOLOGICAL MODEL

In this Section we shall use the field-parametrisation dependence of Dirac’s method to investigate the canonical (natural) parametrisation of the so-called isotropic cosmological model (ICM) (its physical meaning, if any, is not the subject of our discussion). This model is an example of a gauge theory where the Dirac procedure reaches closure on the secondary first-class constraint, unlike in the model of the previous Section. The model is described by the following Lagrangian with two variables  $(N, a)$  [7],

$$L_1 = -\frac{1}{2} \frac{a \dot{a}^2}{N} + \frac{1}{2} N a. \quad (37)$$

Its non-linearity<sup>5</sup> cannot be eliminated by any field redefinition, and the variation of  $L_1$  under the gauge transformations produces a total time derivative, rather than being exactly equal to zero, as is the case of the HTZ model (see (9)). The model represented by Lagrangian (37) attracts considerable attention, and it is a topic of extensive discussion by Shestakova [7, 26, 27]; the choice of a “correct” symmetry for this model [7] (see also [8]) was used to support the claim that “we cannot consider the Dirac approach fundamental and undoubted” [7]. A brief discussion of some aspects of the results in [7] was given in the conclusion of [22] and in our comment [8], which provides a description of a method to find the simplest and most natural parametrisation with the simplest commutator of the gauge transformations, i.e. the *correct* symmetry, without invoking unjustifiable arguments and approximations that are external to the model. We shall use this model to show how a parametrisation, which leads to the simplest symmetry and field variables, can be found in Dirac’s approach.

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<sup>5</sup> The description of non-linearity in [7] was made in reference to the equations; i.e. at the Lagrangian level at least some terms are not quadratic in fields.

For the two parametrisations,

$$N = \sqrt{\mu} \quad (38)$$

(see [7, 8]) and

$$N = e^{-\varkappa} \quad (39)$$

(see [8]), the Dirac procedure yields distinct gauge transformations that belong to the Lagrangian in all parametrisations. These gauge transformations are more complicated than the ones for the original parametrisation (37); and this confirms the conclusion that one may draw from an analysis of (37) – the need for such changes is suspect and unlikely to lead to some simplification<sup>6</sup>. The same conclusion was drawn for the model studied in the previous Section. In [7], the author used the field-parametrisation dependence of the Dirac method to show that it produces the transformations that were proclaimed ‘correct’; but because in another parametrisation the Dirac method did not lead to the same ‘correct’ transformations, it was declared that “we cannot consider the Dirac approach as fundamental and undoubted” [7].

### A. The Hamiltonian and Lagrangian Analyses of ICM

For our discussion, we need the results of the Hamiltonian analysis of (37), which we shall briefly describe (see also [7, 8])). Performing the Legendre transformation

$$H_T^{(1)} = \pi \dot{N} + p \dot{a} - L_1 ,$$

and eliminating the velocities, one obtains

$$H_T^{(1)} = \pi \dot{N} - \frac{1}{2} \frac{N}{a} p^2 - \frac{1}{2} N a. \quad (40)$$

The time development of the primary constraint leads to the secondary constraint  $T^{(1)}$

$$\left\{ \pi, H_T^{(1)} \right\} = \frac{1}{2a} p^2 + \frac{1}{2} a = T^{(1)};$$

and because  $\dot{T}^{(1)} = \{T^{(1)}, H_T\} = 0$ , closure is reached. The total Hamiltonian can then be presented in compact form,

$$H_T^{(1)} = \pi \dot{N} - N T^{(1)}.$$

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<sup>6</sup> Such changes, (38) and (39), convert both terms in the Lagrangian into non-quadratic expressions (actually they both become non-polynomial), contrary to formulation (37).

The algebra of constraints is simple,

$$\{\pi, \pi\} = \{T^{(1)}, T^{(1)}\} = \{\pi, T^{(1)}\} = 0.$$

Using the Castellani procedure [11], the gauge generator can be constructed [7],

$$G^{(1)} = -T^{(1)}\theta_1 + \pi\dot{\theta}_1,$$

which yields the Hamiltonian gauge transformations in phase space. After substitution of the momenta in terms of velocities, it then leads to the Lagrangian form of the gauge transformations<sup>7</sup>,

$$\delta_1 N = \dot{\theta}_1, \quad \delta_1 a = \frac{\dot{a}}{N}\theta_1. \quad (41)$$

The commutator of these transformations is of the simplest possible form,

$$[\delta'_1, \delta''_1] \begin{pmatrix} N \\ a \end{pmatrix} = (\delta'_1 \delta''_1 - \delta''_1 \delta'_1) \begin{pmatrix} N \\ a \end{pmatrix} = 0. \quad (42)$$

As in the previous Section, the DI that follows from transformations (41) is

$$I^{(1)} = -\dot{E}_N^{(1)} + \frac{\dot{a}}{N}E_a^{(1)} \equiv 0, \quad (43)$$

where the ELDs of (37) are:

$$E_N^{(1)} = \frac{\delta L_1}{\delta N} = +\frac{1}{2}\frac{a\dot{a}^2}{N^2} + \frac{1}{2}a \quad (44)$$

and

$$E_a^{(1)} = \frac{\delta L_1}{\delta a} = \frac{a\ddot{a}}{N} + \frac{1}{2}\frac{\dot{a}^2}{N} - \frac{a\dot{a}}{N^2}\dot{N} + \frac{1}{2}N. \quad (45)$$

DI (43) can be checked by the direct substitution of (44) and (45).

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<sup>7</sup> The author of [7] used the generator to find the transformations in the following form  $\delta field = \{field, G\}$ , which gives a minus sign in the gauge transformations; to be consistent with the standard form of a DI, a plus sign is required. The convention of [7] does not affect the results; we can change the sign in DI or incorporate it into a gauge parameter without making it field-dependent, therefore, the commutator of two transformations is unchanged.

## B. Another Choice of Parametrisation

By using a general procedure, let us show for this model how the parametrisation, which leads to the simplest symmetry, can be found in the Hamiltonian approach, and how the natural parametrisation for Lagrangian (37) will emerge from it. One can start from the parametrisations we have already considered, (38) and (39); but we prefer to explore a new parametrisation on the basis of the following observation: in terms of simplicity, there is but one parametrisation for this Lagrangian<sup>8</sup>, i.e.

$$N = \frac{1}{M}. \quad (46)$$

Let us use this parametrisation as a starting point, in an illustrative example of a general procedure for finding a canonical parametrisation. Changing the variables in (37), we obtain the Lagrangian,  $L_4$ , in a new parametrisation<sup>9</sup>,

$$L_4 = -\frac{1}{2}a\dot{a}^2M + \frac{1}{2}\frac{a}{M}. \quad (47)$$

Were we to investigate this Lagrangian, with no foreknowledge of its characteristics, we would apply Dirac's algorithm to it, and find its gauge symmetry by repeating the standard steps that were discussed in detail in [7] for this model. Going to the Hamiltonian, the Legendre transformation yields,

$$H_T^{(4)} = \pi_M \dot{M} + p\dot{a} - L_4 ; \quad (48)$$

one may then find the primary constraint,

$$\pi_M = \frac{\delta L_4}{\delta \dot{M}} = 0, \quad (49)$$

and the generalised momentum,

$$p = \frac{\delta L_4}{\delta \dot{a}} = -a\dot{a}M. \quad (50)$$

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<sup>8</sup> Strictly speaking, this change of variables has a disadvantage – in the original Lagrangian, one term is quadratic in fields and another is ‘non-linear’; with this substitution, both are ‘non-linear’, as was the case in the two parametrisations (38) and (39) considered before [7, 8].

<sup>9</sup> We use the subscript “4” for convenience (in case if one wishes to compare it with [7, 8] where  $L_2(\mu, a)$  and  $L_3(\varkappa, a)$  have been discussed).

From (50), the velocity  $\dot{a}$  can be expressed in terms of momentum  $p$ :

$$\dot{a} = -\frac{p}{aM}. \quad (51)$$

The substitution of (51) into (48) yields the total Hamiltonian,

$$H_T^{(4)} = \pi_M \dot{M} - \frac{1}{2} \frac{p^2}{aM} - \frac{1}{2} \frac{a}{M}. \quad (52)$$

According to the Dirac procedure, one must consider the time development of the primary constraints (i.e. consistency condition)

$$\dot{\pi}_M = \{\pi_M, H_T\} = -\frac{1}{2} \frac{p^2}{aM^2} - \frac{1}{2} \frac{a}{M^2} = T^{(4)},$$

until the closure. Because the time development of  $T^{(4)}$  does not produce new constraints, closure is reached, and the total Hamiltonian can be written in compact form,

$$H_T^{(4)} = \pi_M \dot{M} + MT^{(4)}.$$

The algebra of constraints is simple, and demonstrates that the constraints are first-class:

$$\{\pi_M, \pi_M\} = 0, \quad \{T^{(4)}, T^{(4)}\} = 0, \quad \{\pi_M, T^{(4)}\} = \frac{2}{M} T^{(4)}. \quad (53)$$

This knowledge of first-class constraints is enough for us to find the gauge transformations using the Castellani procedure [11] (which is a formal implementation of Dirac's conjecture [1]), and to construct the generator,

$$G_4 = \left( -\frac{2}{M} \dot{M} \pi_M - T^{(4)} \right) \theta_4 + \pi_M \dot{\theta}_4, \quad (54)$$

in a manner analogous to that used in [7] for  $L_1$  and  $L_2$ , and for  $L_3$  in [8]. The generator allows one to calculate the gauge transformations of the phase-space variables (see footnote 7 about the convention), and also to find the Lagrangian transformations, after substituting velocities in terms of momenta using (51):

$$\delta_4 M = \{M, G_4\} = -\frac{2}{M} \dot{M} \theta_4 + \dot{\theta}_4 \quad (55)$$

and

$$\delta_4 a = \{a, G_4\} = \left\{ a, \frac{1}{2} \frac{p^2}{aM^2} \right\} = \frac{p}{aM^2} = \frac{-a\dot{a}M}{aM^2} = -\frac{\dot{a}}{M} \theta_4. \quad (56)$$

One can check that this is a symmetry of  $L_4$  by directly performing a variation of the Lagrangian:

$$\delta_4 L_4 = \frac{1}{2} \frac{d}{dt} \left( a \dot{a}^2 \theta - \frac{a \theta}{M^2} \right).$$

Alternatively, one may find the Noether DI that corresponds to transformations (55) and (56),

$$I^{(4)} = -\dot{E}_M^{(4)} - \frac{2}{M} \dot{M} E_M^{(4)} - \frac{\dot{a}}{M} E_a^{(4)} \equiv 0, \quad (57)$$

as was done in the previous Section, where the ELDs were calculated for  $L_4$ . DI (57) may be directly confirmed by the substitution of the ELDs.

The commutator of transformations (55)-(56) is

$$[\delta'_4, \delta''_4] \begin{pmatrix} M \\ a \end{pmatrix} = (\delta'_4 \delta''_4 - \delta''_4 \delta'_4) \begin{pmatrix} M \\ a \end{pmatrix} = \delta_4''' \begin{pmatrix} M \\ a \end{pmatrix} \quad (58)$$

with a new gauge parameter

$$\theta_{[\delta'_4, \delta''_4]} = 2 \frac{1}{M} \left( \theta_4'' \dot{\theta}'_4 - \theta_4' \dot{\theta}''_4 \right). \quad (59)$$

The commutator (58) is non-zero, and the new gauge parameter is field-dependent; we must calculate the double commutators to determine whether or not the gauge transformations form an algebra. For such a simple model (as in the previous Section) it is natural to expect that some simpler symmetries and corresponding parametrisations are possible, and that the Hamiltonian can be used to find them. Can we have gauge transformations with a zero commutator, or at least a gauge parameter of the commutator (see (59)) without field dependence? Unlike the model in the previous Section (see DI (17)), the structure of DI (57) is not simple enough to allow one to immediately see what kind of manipulations (if any) can lead to a commutator without field dependence in a new gauge parameter.

### C. Finding the Simplest Gauge Symmetry

Let us seek the simplest symmetry by a general method. Because the leading ELD and the additional contributions that involve this ELD, depend only on the same variable, we may obtain a new DI by performing a multiplication of (57) by some general, unspecified function of  $M$ , i.e.

$$f(M) I^{(4)} = \tilde{I}^{(4)}. \quad (60)$$

We have modified the DI, as well as the corresponding gauge symmetry, in the hope of finding the best parametrisation.

The  $\tilde{I}^{(4)}$  is also a DI, and we can calculate the corresponding transformations (N.B. We keep the parametrisation the same) by the converse of Noether's second theorem [5]. We then try to find the condition on this unspecified function that leads to a zero commutator for these transformations. The transformations for (60) are:

$$\tilde{\delta}_4 M = \frac{d}{dt} (\eta f(M)) - \frac{2}{M} \dot{M} \eta f(M)$$

and

$$\tilde{\delta}_4 a = -\frac{\dot{a}}{M} \eta f(M)$$

(these are different transformations, which we shall call  $\tilde{\delta}_4$ , and we use a new parameter,  $\eta$ ).

For field  $a$  we obtain

$$\left( \tilde{\delta}'_4 \tilde{\delta}_4'' - \tilde{\delta}_4'' \tilde{\delta}'_4 \right) a = \frac{\dot{a}}{M} f(M) (\eta'' \dot{\eta}' - \eta' \dot{\eta}'') \left( -\frac{df}{dM} + 2f(M) \frac{1}{M} \right).$$

And for the commutator to equal zero, we must solve the ODE

$$-\frac{df}{dM} + 2f(M) \frac{1}{M} = 0. \quad (61)$$

The solutions of (61) are

$$f = \pm M^2; \quad (62)$$

and with these values we obtain a new DI and new symmetry in the same parametrisation (we consider the minus sign)<sup>10</sup>:

$$I^{(4)} = M^2 \dot{E}_M + 2M \dot{M} E_M + \dot{a} M E_a \equiv 0.$$

For this DI, the transformations are:

$$\tilde{\delta}_4 M = -M^2 \dot{\eta}, \quad \tilde{\delta}_4 a = \dot{a} M \eta; \quad (63)$$

and this result immediately suggests a simpler form of (60):

$$\tilde{I}^{(4)} = \frac{d}{dt} (M^2 E_M) + \dot{a} M E_a. \quad (64)$$

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<sup>10</sup> Had the plus sign been used instead, then one would have obtained

$$\tilde{I}^{(4)} = -M^2 \dot{E}_M - 2M \dot{M} E_M - \dot{a} M E_a = -I^{(4)} \equiv 0.$$

Therefore the gauge transformations would only differ by a sign.

Is it possible to keep this DI and the corresponding gauge symmetry, but using a different parametrisation convert it to a form that can be obtained in the Hamiltonian approach? In the leading term of (64), we have a field and the corresponding ELD under the sign of the derivative; as before (see previous Section), the search for the needed parametrisation is simplified. Considering a change of only one variable,

$$M = M(\tilde{M}),$$

the following relation among the ELDs can be obtained,

$$E_{\tilde{M}} = E_M \frac{dM}{d\tilde{M}}.$$

Compare it with the first term in (64); the condition to have  $\dot{E}_{\tilde{M}}$  only, in the new variables (i.e. without field-dependent coefficients) is

$$\frac{dM}{d\tilde{M}} = -M^2.$$

Solving this ODE, we obtain,

$$\tilde{M} = \frac{1}{M};$$

thus returning to the original parametrisation (37), which is found to be the best, natural choice for this Lagrangian. We found a parametrisation in which the commutator of transformations is the simplest; therefore, these variables are canonical, and they lead to a canonical Hamiltonian formulation for this model.

#### D. Change of Non-Primary Variables

Does any freedom remain in how to represent the simplest symmetry in a way that is consistent in both the Lagrangian and Hamiltonian approaches? Let us return to Lagrangian (37), which is written in a canonical parametrisation. To have a possibility to obtain this symmetry in the Hamiltonian approach we should not change the variable  $N$ ; to do so would lead to the appearance of a coefficient for the leading ELD in the DI that describes this symmetry (see (43)), and the Hamiltonian would not reproduce it. (The Hamiltonian analysis demonstrates that such a reparametrisation is pathological; the method of finding the canonical parametrisation was illustrated above.)

The only change that preserves the leading ELD is a change of the remaining, non-primary variables (as in the previous Section); and in Lagrangian (37) there is only one:  $a$ . Let us analyse such a change of variable (at the Lagrangian level, it keeps the DI the same, preserving the symmetry; there is no multiplication of the DI by some field-dependent function).

Consider the following general change of the non-primary variable  $a$ ,

$$a = f(b), \quad (65)$$

with the condition that this transformation is invertible. This change only affects a non-leading ELD,

$$E_b^{(1)} = \frac{\delta L}{\delta b} = \frac{\delta L}{\delta a} \frac{\partial a}{\partial b} = E_a^{(1)} \frac{\partial f}{\partial b}. \quad (66)$$

For DI (43), one may re-express the field coefficient in terms of the new variable to obtain,

$$-\dot{E}_N^{(1)} + \frac{\dot{a}}{N} E_{(a)}^{(1)} = -\dot{E}_N^{(1)} + \frac{1}{N} \frac{\partial f}{\partial b} \dot{b} E_{(a)}^{(1)},$$

which leads to a new form of the DI,

$$-\dot{E}_N^{(1)} + \frac{1}{N} \dot{b} E_{(b)}^{(1)} \equiv 0,$$

where  $E_{(b)}^{(1)}$  is taken from (66).

In this case, even the form of the DI is preserved (compare with the previous Section where a general change of two non-primary variables was considered), and the gauge transformations are:

$$\delta_1 N = \dot{\theta}_1, \quad \delta_1 b = \frac{\dot{b}}{N} \theta_1.$$

Is it the same symmetry? Using field redefinition (65) we find

$$\delta_1 a = \frac{\delta f}{\delta b} \delta_1 b = \frac{\delta f}{\delta b} \frac{\dot{b}}{N} \theta_1 = \frac{\dot{a}}{N} \theta_1,$$

where  $\dot{a} = \frac{\partial f}{\partial b} \dot{b}$  was used. So any invertible change of non-primary variables is permissible; it does not affect the gauge symmetry and the gauge transformation remains the same.

We can start from the Lagrangian and obtain the same result by performing the Hamiltonian analysis; these are simple calculations, which are similar to those we have already presented in this paper. We shall not repeat them. We wish to mention that this change,

(65), which is similar to (29) described in the previous Section, can be made at the level of the total Hamiltonian by using a canonical change of phase-space variables

$$p_a \dot{a} = p_a \frac{\delta f}{\delta b} \dot{b} = p_b \dot{b}, \quad p_b = p_a \frac{\delta f}{\delta b}; \quad (67)$$

the formulae in (67) preserve the algebra of the first-class constraints, as was conjectured in [22] and illustrated for the slightly more complicated change of two variables in the previous Section (see (29)).

The change of variable  $N$ , which does offer any improvement for the few parametrisations that were considered (see (38), (39), and (46)), can be repeated in a general form in the Hamiltonian and Lagrangian approaches, to show that only one parametrisation leads to a zero commutator. The change of  $a$  does not affect the gauge symmetry, and even the form of the DI is preserved.

For the singular Lagrangian (37), its original parametrisation is the most natural; it has a simple Noether DI, where the coefficient of the leading ELD (with the highest order derivative) is not field dependent, and the commutator of the transformations is the simplest possible (see (42)).

### E. Change of Parametrisation – Mixture of Primary and Non-Primary Variables

For completeness, let us consider a change of variables that uses a mixture of  $N$  and  $a$ . Great attention was paid to such changes in the papers of Shestakova [7, 26–28] as a prerequisite to the attempt to “restore a legitimate status of the ADM parametrisation” [7] as a representation of the Hamiltonian formulation of metric GR. Let us consider the effect of such a change in this simple model; but instead of the general form  $N = v(\tilde{N}, a)$  of [7], we perform a simple change,

$$N = \frac{a}{K}. \quad (68)$$

This change of variables was selected in [7] to simplify the Lagrangian:

$$L_5(K, a) = -\frac{1}{2} K \dot{a}^2 + \frac{1}{2} \frac{a^2}{K}. \quad (69)$$

Of course any invertible change of fields in the Lagrangian preserves its symmetry; this fact can be confirmed by using change of variables (68) and recalculating the DI that describes the simplest transformations with a zero commutator. A DI similar to (43) can be

found for Lagrangian (69). To obtain a new form of DI, we need a relation between the ELDs for both of the variables that follow from

$$\delta L_1 = E_N^{(1)} \delta N + E_a^{(1)} \delta a = E_K^{(5)} \delta K + E_a^{(5)} \delta a = \delta L_5 . \quad (70)$$

Using (68), we have

$$\delta K = -\frac{a}{N^2} \delta N + \frac{1}{N} \delta a = -\frac{K^2}{a} \delta N + \frac{K}{a} \delta a,$$

which after performing a substitution into (70) and making a comparison of the coefficients of  $\delta N$  and  $\delta a$  yields

$$E_N^{(1)} = -\frac{K^2}{a} E_K^{(5)}$$

and

$$E_a^{(1)} = \frac{K}{a} E_K^{(5)} + E_a^{(5)}.$$

Substitution of these results into (43) leads to a new form of DI,

$$I^{(1)}(K, a) = -\frac{d}{dt} \left( -E_K^{(5)} \frac{K^2}{a} \right) + \frac{\dot{a}K}{a} \left( \frac{K}{a} E_K^{(5)} + E_a^{(5)} \right) \equiv 0.$$

Its simplification gives

$$I^{(1)}(K, a) = \frac{K^2}{a} \dot{E}_K^{(5)} + \frac{2K}{a} \dot{K} E_K^{(5)} + \frac{\dot{a}K}{a} E_a^{(5)} \equiv 0, \quad (71)$$

which corresponds to the following gauge transformations:

$$\delta_1 K = -\frac{K^2}{a} \dot{\theta}_1 + \frac{K^2}{a^2} \dot{a} \theta_1, \quad \delta_1 a = \frac{\dot{a}K}{a} \theta_1 . \quad (72)$$

These are the same transformations as  $\delta_1$  in (41), but written for field parametrisation (68) instead. The commutator of two such transformations is zero, as before (although it is not as obvious as it was for (41), where in  $\delta_1 N$  only a gauge parameter is present). The DI has a field-dependent coefficient in front of the leading ELD; and in this parametrisation, the Hamiltonian cannot reproduce this symmetry.

We now wonder what symmetry would follow from the Dirac procedure. Starting from Lagrangian (69), let us find the Hamiltonian and restore the gauge transformations. Performing the Legendre transformation yields

$$H_T^{(5)} = \tilde{\pi} \dot{K} + p \dot{a} - L_5 ,$$

where the primary constraint is

$$\tilde{\pi} = 0.$$

By eliminating the velocity in terms of a momentum,

$$p = \frac{\delta L_5}{\delta \dot{a}} = -K\dot{a}, \quad \dot{a} = -\frac{p}{K}, \quad (73)$$

we obtain

$$H_T = \tilde{\pi}\dot{K} - \frac{1}{2K} (p^2 + a^2). \quad (74)$$

The time development of the primary constraint is

$$\{\tilde{\pi}, H_T\} = -\frac{1}{2K^2} (p^2 + a^2) = \chi,$$

and the time derivative of the secondary constraint gives

$$\{\chi, H_T\} = \left\{ \chi, \tilde{\pi}\dot{K} \right\} = -2\frac{\dot{K}}{K}\chi. \quad (75)$$

Therefore the procedure is closed on the secondary constraint. The Castellani generator is

$$G = G_{(1)}\dot{\theta}_5 + G_{(0)}\theta_5,$$

where

$$G_{(1)} = \tilde{\pi},$$

and

$$G_{(0)} = -\{G_{(1)}, H_T\} + \alpha\tilde{\pi}.$$

The function  $\alpha$  can be found by using the condition,

$$\{G_{(0)}, H_T\} = \text{primary}.$$

Calculations similar to those presented in Appendix A yield  $\alpha = -2\frac{\dot{K}}{K}$ ; and the explicit form of the generator is

$$G = \tilde{\pi}\dot{\theta}_5 + \left( -\chi - 2\frac{\dot{K}}{K}\tilde{\pi} \right) \theta_5, \quad (76)$$

which results in the following gauge transformations:

$$\delta K = -\dot{\theta}_5 + 2\frac{\dot{K}}{K}\theta_5, \quad \delta a = \frac{\dot{a}}{K}\theta_5. \quad (77)$$

Note that  $\delta a = \{G, a\}$  gives an expression that depends on a momentum, which can be eliminated using (73) to obtain the transformations in configurational space for the Lagrangian.

The DI that corresponds to these transformations is

$$I^{(5)} = \dot{E}_K^{(5)} + 2\frac{\dot{K}}{K}E_K^{(5)} + \frac{\dot{a}}{K}E_a^{(5)} \equiv 0;$$

and there is no need to check that this is a DI because it is related to  $I^{(1)}$  from (71) by

$$I^{(5)} = \frac{a}{K^2} I^{(1)}(K, a) .$$

We note that if the commutator of the transformations was zero for the natural parametrisation of this model, then in a new parametrisation it is

$$[\delta'_5, \delta''_5] \begin{pmatrix} \tilde{N} \\ a \end{pmatrix} = \delta'''_5 \begin{pmatrix} \tilde{N} \\ a \end{pmatrix} ,$$

with the following gauge parameter

$$\theta_{[\delta'_5, \delta''_5]} = \frac{2}{K} \left( \dot{\theta}'_5 \theta''_5 - \dot{\theta}''_5 \theta'_5 \right) ,$$

which unlike (42), is now field- dependent.

## F. Mixture of Primary and Non-primary Variables at the Hamiltonian Level

Finally, let us try to apply a change of variables (68) directly at the Hamiltonian level. Hamiltonians (40) and (74) cannot be canonically related because they describe different symmetries; therefore, the algebras of constraints are different (see (53) and (75)). But another problem arises even in this simple model if a change of variables mixes the primary with non-primary variables. To perform a substitution in the Hamiltonian, we have to find the canonical transformations in phase space. We consider

$$\begin{aligned} \pi_N \dot{N} + p \dot{a} &= \pi_N \left( -\frac{a}{K^2} \dot{K} + \frac{1}{K} \dot{a} \right) + p \dot{a} = \\ &= -\pi_N \frac{a}{K^2} \dot{K} + \left( \pi_N \frac{1}{K} + p \right) \dot{a}, \end{aligned}$$

which leads to the following redefinition of momenta:

$$\pi_N = -\pi_K \frac{K^2}{a} \tag{78}$$

and

$$p = \tilde{p} - \pi_N \frac{1}{K} = \tilde{p} + \pi_K \frac{K}{a}. \tag{79}$$

This redefinition automatically preserves the canonical PBs among the two sets of phase-space variables (as in (35) of the previous Section). But the substitution of such transformations, (78) and (79), into the total Hamiltonian introduces momenta, conjugate to the primary variable, into the canonical part of the Hamiltonian,

$$H_T^{(1)} = -\pi_K \dot{K} - \pi_K \frac{K}{a} \dot{a} - \frac{1}{2K} \left( \tilde{p} + \pi_K \frac{K}{a} \right)^2 - \frac{1}{2} \frac{a^2}{K}; \quad (80)$$

further, the time derivatives of the variables, which had been eliminated, reappear, illustrating the collapse of the Hamiltonian formulation. Note that similar problems occurred in more complicated cases, e.g. for the change of the original variables of GR, the components of the metric tensor, to the ADM variables, which cannot be performed in the total Hamiltonian [2]. In the original, natural parametrisation, the Hamiltonian formulation leads to diffeomorphism in configurational space [3, 4], and for the ADM parametrisation it produces a different symmetry that does not form a group [8, 9]. The change of variables due to ADM, is a mixture of the primary with non-primary variables (see [2] and [13]).

For a change of phase-space variables, which involves a mixture of primary and non-primary variables as in (68), (78), and (79), even having canonical PBs does not preserve the equivalence of the two Hamiltonians, and some nonsensical results are obtained (see (80)). Even these simple models show that such a change of variables does not to preserve a gauge symmetry. Yet on the basis of such manipulations, some have concluded that “clear proof” of the legitimacy of the ADM variables is given [7]. For covariant theories (at least those that are covariant before some non-covariant parametrisations are introduced) the effect of the field-parametrisation dependence of the Dirac method is left for future discussion.

#### IV. CONCLUSION

The Euler-Lagrange derivatives of gauge invariant Lagrangians are not independent; they can be combined into linear combinations, i.e. differential identities, each of which is identically equal to zero (“off-shell”). The important characteristic of a gauge invariant Lagrangian is the maximum number of independent DIs, which is equal to the number of gauge parameters [5]; but because any combination of DIs can be constructed, the DIs cannot be uniquely specified. It is possible for a set of DIs, which describes a particular symmetry, to be written in different field parametrisations. Alternatively, different DIs might correspond to the gauge

transformations, which may or may not have group properties (e.g. see [9, 24]); and even with group properties, commutators of gauge transformations may be of varying complexity (e.g. compare (15)-(16) with (20), and (42) with (58)-(59); see also [8]).

For covariant theories, in which natural field parametrisations (where fields are true tensors) are used, the ELDs, DIs, and corresponding gauge transformations are also automatically true tensors (or tensor densities). Therefore, all of them are independent of the choice of coordinates when written in explicitly covariant form (e.g. if a DI is a vector density, as in metric GR, it is identically zero in all coordinate systems). Any non-covariant modifications of ELDs, or the use of non-covariant field parametrisations, destroys such properties; hence, severe restrictions on coordinate transformations are required to preserve the validity of ELDs, DIs, and gauge transformations. In covariant theories the natural restriction of the results to covariant form leads to covariant gauge transformations; when supplemented by the requirement that the DIs be of the lowest order in the derivatives of ELDs, the result is unique (although from a covariant DI, an additional covariant DI with the highest order of derivatives can be constructed, as was demonstrated in the conclusion of [9]). The Hamiltonian formulation, when based on a Lagrangian written in a natural parametrisation (covariant variables for covariant Lagrangians), produces covariant gauge transformations for all variables, as one may return to the gauge transformations in configurational space. Different parametrisations in the Hamiltonian approach lead to different symmetries, which in complicated theories (without reference to covariance) can lose the group properties of the transformations, or possess group properties but with a commutator of gauge transformations which has field-dependent structure functions. This is how the Hamiltonian formulation (which is not covariant by construction) can be used to find the preferable parametrisation of a covariant theory, based on the simplest gauge transformation properties, upon which parametrisation is found to be covariant. In fact, the field-parametrisation dependence of the Dirac method offers a way to preserve covariance when using the Hamiltonian method, which is innately non-covariant.

For non-covariant models, the guidance provided by covariance on what choice of variables is natural, is absent; therefore, the field-parametrisation dependence of the Dirac procedure becomes more important, since it allows one to find the natural variables for the Lagrangian being considered (i.e. variables in which the Hamiltonian formulation leads to the gauge transformations with the simplest algebra). We have used two examples (see Sections 2 and

3) to illustrate how to find the natural variables. The procedure can be described as follows: start with a Lagrangian, which is written in some original parametrisation, and pass to its Hamiltonian formulation; find the gauge transformations; use Noether’s second theorem [5] to find the corresponding DIs; modify these DIs (while still in the original parametrisation) by multiplying them by an unspecified function of field variables; and use the converse of Noether’s theorem to find new gauge transformations that correspond to a new DI; then calculate the commutator; try to specify the function of field variables by imposing the condition that the commutator be zero; if it is not possible to do so, then seek a new field-independent structure constant; if this search fails, then find a new commutator with a structure function that depends on fields, but ensure that the gauge transformations form a group (the calculation of a double commutator and Jacobi identity, as in [9, 24], might be required).

Modified DIs, that are not of a simple form (e.g. DI (17)), and the corresponding gauge transformations cannot be obtained through the Hamiltonian approach; but by performing a change of field variables, a complicated DI can be simplified (i.e. the coefficient of the leading ELD can be made field-independent). The corresponding parametrisation is then natural for the model being considered. Further, the new gauge transformations can also be obtained in the Hamiltonian approach. In the two examples treated in this paper, we explicitly demonstrated this procedure; but it can be applied to any model, and thus it becomes a more technically involved consideration (e.g. in the ADM parametrisation the Hamiltonian approach yields transformations without group properties; it would be a Herculean Labour to try to find a change of variables to bring one back to the natural parametrisation of the Einstein-Hilbert action, i.e. the metric tensor, for which the Hamiltonian formulation produces gauge transformations with group properties).

The important role of the “primary” variables was demonstrated in [13, 25] (at the Lagrangian level, primary variables are those for which the corresponding ELDs enter the DIs with the highest order of time derivative, e.g. see (17)). Any change of fields that involves the primary variables profoundly affects the properties of the gauge transformations, unlike changes that only involve the non-primary variables (in [13, 25] they were called “non-primary”, “secondary” or “second class”). Any canonical change of secondary variables does not affect the group properties of the commutators, this behaviour is also observed in non-singular theories and models (where all variables are actually secondary).

The primary variables differ greatly from non-primary ones, and ordinary canonicity [21] for the change of variables is a necessary, but not a sufficient condition to preserve the symmetry in the two Hamiltonian formulations; the whole algebra of first-class constraints must also be preserved. The need for such requirements was found when making the comparison of the two Hamiltonian formulations of metric GR: PSS [3], and Dirac's [4] – both lead to diffeomorphism in configurational space [2, 23]. Changes that only involve non-primary variables keeps the properties of the commutators in tact (e.g. (65)). A mixing of primary and non-primary variables might drastically affect the result of the Hamiltonian formulation (e.g. see Section 3). There is, however, one special class of field parametrisation – a rescaling of primary variables by functions of non-primary variables, for example, for the models considered in Sections 2 and 3, it would be  $\tilde{q} = qf(q_1, q_2)$  and  $\tilde{N} = Nf(a)$ , respectively.

Consider the natural parametrisation of (24), for which the commutator of the gauge transformations is zero, such rescaling will not change the commutator, although the gauge transformations of the new fields will be different. Therefore, with this additional freedom, the effect of various parametrisations for the Hamiltonian formulation allows one to find the natural parametrisation (with the simplest commutator) only up to such a rescaling. This additional freedom could possibly be related to so-called counterexamples to the Dirac conjecture that are discussed in the literature. Some counterexamples just result from incorrect Hamiltonian and Lagrangian analyses (e.g. [29]), while in some papers, Dirac's conjecture is defended [30–32]. Some examples are more complicated, such as the widely known Cawley Lagrangian [33], which could just be a consequence of using an unnatural parametrisation for the proposed model. There could be parametrisations that not only do not lead to a gauge symmetry with the simplest commutator, but even make the application of the Dirac procedure impossible. The solution to this problem is to consider different parametrisations, including the rescaling of primary variables. But it happens that all of the counterexamples just mentioned can be resolved by a better choice of parametrisation; or equally well, that all such counterexamples can be seen as some models that are broken by a poor choice of parametrisation. Further, the models with an occurrence of a square of constraints (e.g. [33]) were excluded by Castellani in his theorem (in which he proved the Dirac conjecture), i.e. “*all the FC [first-class constraints], except those arising as  $\chi^n \dots$ , are part of gauge generators...*” [11] (p. 364).

The first-class constraints of the Hamiltonian formulation and the DIs in the Lagrangian

formulation are interrelated, also the gauge generators are linear in constraints and the DIs are linear combinations of ELDs; therefore, the appearance of a square or higher power of constraints in the Hamiltonian analysis of a model is a strong indication of the need to change a parametrisation. The role of field parametrisation in the counterexamples to the Dirac conjecture is one aspect of our current investigation, and the results will be reported elsewhere.

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### Appendix A: The Henneaux-Teitelboim-Zanelli model using the Castellani procedure

Let us first outline the steps to be followed in performing the Castellani procedure [11]. We have already used this procedure to restore the gauge invariance of various Hamiltonian formulations (e.g. [2, 13, 23, 25, 34–36]). In [13] we describe the peculiarities that arise when the Castellani procedure is applied to systems with tertiary constraints, which is also important for the system under consideration – the HTZ model.

The generator of gauge transformations for the Hamiltonian, with first-class constraints for a system where tertiary constraints appear, is given by

$$G(t) = \varepsilon G_{(0)} + \dot{\varepsilon} G_{(1)} + \ddot{\varepsilon} G_{(2)} , \quad (\text{A1})$$

where  $\varepsilon$  is the gauge parameter, and  $\dot{\varepsilon}$  and  $\ddot{\varepsilon}$  are its first and second derivatives with respect to time (as in the HTZ model,  $\varepsilon$  and  $G_{(i)}$  are functions of time only). The number of gauge parameters and their tensorial dimension (for covariant theories) are uniquely defined by the number of primary first-class constraints, so for the formulation considered (HTZ), there is one gauge parameter,  $\varepsilon$ . The functions  $G_{(i)}$  can be found through the following iterative procedure (see Eq. (16b) [11], and for more details of its application to field theory see also Sections 5 and 6 of [11]):

$$G_{(2)} = \phi_1 , \quad (\text{A2})$$

$$G_{(1)} + \{G_{(2)}, H_T\} = \alpha\phi_1 , \quad (\text{A3})$$

$$G_{(0)} + \{G_{(1)}, H_T\} = \beta\phi_1 , \quad (\text{A4})$$

$$\{G_{(0)}, H_T\} = \text{primary} . \quad (\text{A5})$$

(Here,  $\alpha$  and  $\beta$  are functions of phase-space variables.) Note that only primary constraints explicitly enter equations (A2)-(A5). The function  $G_{(2)}$  is uniquely defined as a primary constraint  $\phi_1$ , while the functions  $G_{(1)}$  and  $G_{(0)}$  are, in general, not just the secondary or tertiary constraints. When we calculated the tertiary constraint in (5), we only kept the contribution that was not proportional to the constraints found at previous stages in the Dirac procedure. In the Castellani procedure, however, the complete expression for the time development of a constraint is needed. For example, for  $\{\phi_2, H_T\}$  in  $\{G_{(1)}, H_T\}$  and  $\{G_{(0)}, H_T\}$ , we must substitute

$$\{\phi_2, H_T\} = \{-e^{q_1} p_2, H_T\} = \phi_2 \dot{q}_1 + \phi_3 . \quad (\text{A6})$$

Using the PBs among the first-class constraints, and the total Hamiltonian, which are given by (5), (6), (A6), and (2), we can solve (A3) and (A4) for  $G_{(1)}$  and  $G_{(0)}$ , respectively:

$$G_{(1)} = -\{G_{(2)}, H_T\} + \alpha\phi_1 = -\phi_2 + \alpha\phi_1 , \quad (\text{A7})$$

$$G_{(0)} = -\{G_{(1)}, H_T\} + \beta\phi_1 = -\{-\phi_2 + \alpha\phi_1, H_T\} + \beta\phi_1 = \phi_2 \dot{q}_1 + \phi_3 - \alpha\phi_2 + \beta\phi_1 . \quad (\text{A8})$$

The time development of  $G_{(0)}$  from (A5) allows us to find unknown functions  $\alpha$  and  $\beta$ :

$$\begin{aligned} \{G_{(0)}, H_T\} &= \{\phi_3 + \phi_2 \dot{q}_1 - \alpha\phi_2 + \beta\phi_1, H_T\} \\ &= \phi_3 \dot{q}_1 + (\phi_2 \dot{q}_1 + \phi_3) \dot{q}_1 - \phi_2 \ddot{q}_1 - \alpha(\phi_2 \dot{q}_1 + \phi_3) + \beta\phi_2 = \text{primary} . \end{aligned} \quad (\text{A9})$$

Collecting terms with constraints  $\phi_3$  and  $\phi_2$  gives us equations for  $\alpha$  and  $\beta$ :

$$2\dot{q}_1 - \alpha = 0 , \quad (\text{A10})$$

$$(\dot{q}_1)^2 - \ddot{q}_1 - \alpha\dot{q}_1 + \beta = 0 . \quad (\text{A11})$$

Solving (A10)-(A11) yields:

$$\alpha = 2\dot{q}_1 , \quad \beta = (\dot{q}_1)^2 + \ddot{q}_1 . \quad (\text{A12})$$

Finally, the gauge generator (A1), with  $\alpha$  and  $\beta$  from (A12), is

$$G = \varepsilon [\phi_3 - \dot{q}_1 \phi_2 + ((\dot{q}_1)^2 + \ddot{q}_1) \phi_1] + \dot{\varepsilon} (-\phi_2 + 2\dot{q}_1 \phi_1) + \ddot{\varepsilon} \phi_1 . \quad (\text{A13})$$

To obtain the gauge transformations, we use  $\delta q_i = \{G, q_i\}$  and the explicit expressions for constraints  $\phi_i$  (3)-(5):

$$\begin{aligned}\delta q_3 = \{G, q_3\} &= -\frac{\delta G}{\delta p_3} = -\frac{\delta(\varepsilon \phi_3)}{\delta p_3} = -\varepsilon \frac{\delta(e^{q_1} p_3)}{\delta p_3} = -\varepsilon e^{q_1} , \\ \delta q_2 = \{G, q_2\} &= -\frac{\delta G}{\delta p_2} = -\frac{\delta(-\varepsilon \dot{q}_1 \phi_2 - \dot{\varepsilon} \phi_2)}{\delta p_2} = (\varepsilon \dot{q}_1 + \dot{\varepsilon}) \frac{\delta(-e^{q_1} p_2)}{\delta p_2} = -(\varepsilon \dot{q}_1 + \dot{\varepsilon}) e^{q_1} , \\ \delta q_1 = \{G, q_1\} &= -\frac{\delta G}{\delta p_1} = -\frac{\delta(\varepsilon((\dot{q}_1)^2 + \ddot{q}_1)\phi_1 + 2\dot{\varepsilon}\dot{q}_1\phi_1 + \ddot{\varepsilon}\phi_1)}{\delta p_1} \\ &= -\varepsilon((\dot{q}_1)^2 + \ddot{q}_1) - 2\dot{\varepsilon}\dot{q}_1 - \ddot{\varepsilon} .\end{aligned}$$

These results prove that generator (A13) produces gauge transformations (8). There is a difference in sign, but the authors of [10] employed the convention  $\delta q_i = \{q_i, G\}$ . We conclude that there is no advantage in using the HTZ approach for finding gauge transformations as the amount of calculation is the same as that for the Castellani procedure [11].

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